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AUGUST 1965

ALTERNATIVE TECHNIQUES FOR THE INVARIANT IMBEDDING OF RAREFIED COUETTE FLOWS

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PREFACE

Invariant imbedding techniques have proven useful in the computational solution of problems in neutron transport and radiative transfer. This Memorandum discusses an application of these techniques to a problem in rarefied gas dynamics and should be of interest to aerodynamicists, physicists, and applied mathematicians.

SUMMARY

This Memorandum re-examines the application of invariance techniques to plane Couette-flow problems and presents several different approaches to the derivation of the invariant-imbedding equations.

It is shown how both linear and nonlinear collision operators may be considered, for both discrete and continuous versions of the original Boltzmann formulation.

CONTENTS

SYMBOLS

$\vec{C}_i$	=	velocity vector, defined by three velocity components u_{i_u}, v_{i_v}, w_{i_w}
f_i	=	distribution function for particles with velocity vector $\vec{C}_i$
K_i	=	collision operator for particles in i^{th} velocity state
L	=	linearized collision operator
L^+	=	linearized collision operator acting on Φ^+
L^-	=	linearized collision operator acting on Φ^-
R	=	reflection function
R^*	=	nonlinear reflection function
r_i^+, w_i^-	=	defined by Eq. (20)
S	=	one-dimensional reflection function
u, v, w	=	molecular velocities for particles in Φ^+
u_{i_u}, v_{i_v}, w_{i_w}	=	velocity components in discrete velocity space
u_o, v_o, w_o	=	molecular velocities for particles in Φ^-
x	=	distance along normal to a moving plate
z	=	spacing between plates
γ_i^-	=	source function for molecules in i^{th} state entering at x = z
Φ	=	perturbation distribution function
Φ^+	=	perturbation distribution function for particles with velocity components in the direction of increasing x
Φ^-	=	perturbation distribution function for particles with velocity components in the direction of decreasing x

I. INTRODUCTION

In an earlier paper,[1] the concepts of invariant imbedding were applied to a simple problem in rarefied gas dynamics, that of steady, linearized shear flow between two infinite plane walls under the assumptions of Krook scattering and diffuse reflection from both walls. Particle-counting techniques were applied directly to the Krook collision operator, reflection functions were introduced, and the characteristically nonlinear integro-differential equation of the initial-value type for the reflection function was derived. Subsequent work[2] has concentrated on the computational solution of that integral equation as well as on the unsteady version of this same linearized shear flow and on the effect of specular-diffuse molecular reflection from boundaries.

In this Memorandum, we briefly re-examine the application of invariance techniques to plane Couette-flow problems and present several alternate prescriptions for the derivation of the invariant-imbedding equations directly from the Boltzmann formulation. Our objective here is not to present any new results but rather to review and extend several techniques which have already been developed.

It is clear by now that a more formal and mechanical approach than particle counting is desirable for the derivation of invariance equations. Particle counting, unless performed with great care, can lead to errors in problems involving complex geometries and nonlinear scattering models. A rigorous treatment of the equations of invariant imbedding has recently been given by Bailey,[3] who considered a number of related linear problems in transport theory. In particular, he has re-examined the earlier analyses for spherical geometries and concludes that perturbation techniques are the most direct method of approach.

The invariant-imbedding approach involves the calculation of the effect of a variation of the spacing between the two boundaries for a fixed incoming distribution function of a particle at one boundary. We shall consider two different approaches to the problem; the first, that of Wing,[4] is suitable for linear problems, while the second, that of Bellman and Kalaba,[5] is suitable for both linear and nonlinear problems.

II. ANALYSIS

METHOD I

We consider a linearized Couette flow between two infinite planar walls. The Boltzmann equation for this flow becomes

$$u \frac{\partial \Phi}{\partial x} (u, v, w) = L(\Phi) \tag{1}$$

The notation of Ref. 1 is employed, where u, v, and w are dimensionless molecular velocities, Φ is a perturbation distribution function, x is distance in units of mean free path, and $L(\Phi)$ is a linearized collision operator. It was shown in Ref. 2 how it is possible to uncouple the effect of the boundary-surface interaction by dealing with a standard set of boundary conditions corresponding to one wall being a sink for molecules and the other wall being a molecular source. This uncoupling is only valid for linearized flows. The solution of this "standard" problem can then be used to construct solutions which include the effects of nondiffuse surface-particle interaction. The standard boundary conditions are*

$$\Phi^+(u, v, w, f(u_o, v_o, w_o), 0) = 0 \qquad \text{(sink at } x = 0)$$

$$\Phi^-(u_o, v_o, w_o, f(u_o, v_o, w_o), z) = f(u_o, v_o, w_o) \qquad \text{(arbitrary source of molecules at } x = z) \tag{2}$$

where we have chosen to indicate explicitly the dependence of the Φ's on f(u, v, w). We seek a function $R(u, v, w, u_o, v_o, w_o)$, called a reflection function, such that

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*The subscripted variables u_o, v_o, w_o refer to the velocity space of particles with velocity components in the direction of decreasing x, or those particles in Φ^-, while the unsubscripted variables u, v, w refer to the velocity space of particles with velocity components in the direction of increasing x, or those particles in Φ^+.

$$\Phi^{+}(u, v, w, f(u, v, w), z) = \int_{-\infty}^{0} \int_{-\infty}^{-\infty} \int_{-\infty}^{+\infty} R(u, v, w, u_o, v_o, w_o, z)$$

$$f(u_o, v_o, w_o)\, du_o\, dv_o\, dw_o \tag{3}$$

We consider the same problem as that given by Eq. (3) and boundary conditions of Eq. (2) on the interval $0 \leq x \leq z + \Delta$. Then

$$\begin{aligned} \Phi^{+}(u, v, w, f(u_o, v_o, w_o, 0) &= 0 \\ \Phi^{-}(u_o, v_o, w_o, f(u_o, v_o, w_o), z + \Delta) &= f(u_o, v_o, w_o) \end{aligned} \tag{4}$$

and

$$\Phi^{+}(u, v, w, f(u_o, v_o, w_o), z + \Delta) = \int_{-\infty}^{0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R(u, v, w, u_o, v_o, w_o, z + \Delta)$$

$$f(u_o, v_o, w_o)\, du_o\, dv_o\, dw_o \tag{5}$$

From Eq. (1), we can write

$$\Phi^{+}(z + \Delta, f) =$$

$$\Phi^{+}(z, f) + \frac{1}{u} \Delta L^{+}(\Phi^{+}(u, v, w, f, z), \Phi^{-}(u_o, v_o, w_o, f, z) + o(\Delta)) \tag{6}$$

where

$$\lim_{\Delta \to 0} \frac{o(\Delta)}{\Delta} = 0$$

and $f \equiv f(u_o, v_o, w_o)$. We now observe that if we consider this problem on the interval $0 \leq x \leq z$ we may write

$$\Phi^{+}(u, v, w, f, z) = \int_{-\infty}^{0}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R(u, v, w, u_o, v_o, w_o, z)$$

$$f^{*}(u_o, v_o, w_o)\, du_o\, dv_o\, dw_o \tag{7}$$

where $f^{*}(u_o, v_o, w_o)$ is the source function for the new problem.

$$f^{*}(u_o, v_o, w_o) = \Phi^{-}(u_o, v_o, w_o, f, z)$$

$$= \Phi^{-}(u_o, v_o, w_o, f, z + \Delta) - \frac{\Delta}{u_o} L^{-}(z + \Delta) + o(\Delta) \tag{8}$$

$$= f(u_o, v_o, w_o) - \frac{\Delta L^{-}}{u_o} (z + \Delta) + o(\Delta)$$

Thus, from Eqs. (5), (6), and (8), we may write

$$\int_{-\infty}^{0}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R(u, v, w, u_o, v_o, w_o, z + \Delta) f(u_o, v_o, w_o)\, du_o\, dv_o\, dw_o$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} R(u, v, w, u_o, v_o, w_o, z)$$

$$\left\{f(u_o, v_o, w_o) - \frac{\Delta L^{-}}{u_o} (z + \Delta)\right\} \times du_o\, dv_o\, dw_o + \frac{1}{u} \Delta L^{+}(z) + o(\Delta) \tag{9}$$

We note that

$$\Delta L^{\pm}(z + \Delta) = \Delta L^{\pm}(z) + o(\Delta)$$

Equation (9) contains the gist of the invariant-imbedding argument.[†]

The Krook equation possesses the property that the perturbation distribution function and the linearized collision operator are reducible to a dependence on the single velocity variable, u, for linearized

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[†]If the linearized collision operator L is specified, then an integro-differential equation for $R(u, v, w, u_o, v_o, w_o)$ is recovered.

steady and unsteady shear flows (boundary conditions permitting). This, of course, is a major virtue of this scattering model. Consequently, integrations over v_o and w_o are performed explicitly, and a significant reduction in the number of independent variables is obtained. An equation is obtained for the one-dimensional reflection function $R(u, u_o, \tau)$ by combining Eqs. (9) and (10).

$$L^+ \equiv \left(\frac{\delta\Phi^+}{dt}\right)_{collisions} = -\left\{\Phi^+ - \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\,\Phi^+\,du - \frac{1}{\sqrt{\pi}}\int_{-\infty}^0 e^{-u_o^2}\,\Phi^-\,du_o\right\}$$

$$L^- \equiv \left(\frac{\delta\Phi^-}{\delta t}\right)_{collisions} = -\left\{\Phi^- - \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\,\Phi^+\,du - \frac{1}{\sqrt{\pi}}\int_{\infty}^0 e^{-u_o^2}\,\Phi^-\,du_o\right\} \tag{10}$$

Introducing Eq. (10) into Eq. (9), we obtain

$$\int_{-\infty}^0 R(u, \bar{u}_o, z+\Delta) f(\bar{u}_o)\,du_o = \int_{-\infty}^0 R(u, \bar{u}_o, z)\left\{f(\bar{u}_o) + \frac{\Delta}{\bar{u}_o}\right.$$

$$\times\left[f(\bar{u}_o) - \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-u'^2}\int_{-\infty}^0 R(u', u_o', z) f(u_o')\,du_o'\,du'\right]$$

$$\left. - \frac{1}{\sqrt{\pi}}\int_{-\infty}^0 f(u_o') e^{-u_o'^2}\,du_o'\right\} d\bar{u}_o - \frac{1}{\sqrt{\pi}}\frac{1}{u}\Delta\int_{-\infty}^0 R(u, \bar{u}_o, z) f(\bar{u}_o)\,d\bar{u}_o$$

$$+ \frac{1}{\sqrt{\pi}}\frac{1}{u}\Delta\int_0^\infty e^{-u'^2}\int_{-\infty}^0 R(u', u_o', z) f(u_o')\,du_o'\,du'$$

$$+ \frac{1}{\sqrt{\pi}}\frac{1}{u}\Delta\int_{-\infty}^0 f(u_o') e^{-u_o'^2}\,du_o' \tag{11}$$

where $\bar{u}_o$, u_o', and u' are dummy variables.

This relationship must hold for all source functions f(u). We select the particular value corresponding to a Dirac-delta-function source, i.e.,

$$f(u) = \delta(u - u_o)$$

Therefore,

$$R(u, u_o, \tau + \Delta) = R(u, u_o, z) + \frac{\Delta}{u_o} R(u, u_o, z)$$

$$- \Delta \int_{-\infty}^{0} R(u, u_o', z) \cdot \frac{1}{u_o'\sqrt{\pi}} \int_{0}^{\infty} e^{-u'^2} R(u', u_o)\, du'\, du_o'$$

$$- \Delta \int_{-\infty}^{0} R(u, u_o', z) : \frac{1}{\sqrt{\pi}} \frac{1}{u_o'} e^{-u_o^2}\, du_o' - \frac{\Delta}{u} R(u, u_o, z)$$

$$+ \frac{\Delta}{u\sqrt{\pi}} \int_{0}^{\infty} e^{-u'^2} R(u', u_o, z)\, du' + \frac{\Delta}{u\sqrt{\pi}} e^{-u_o^2} \qquad (12)$$

Letting $\Delta \to 0$, and introducing

$$R(u, u_o, z) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{u}} e^{-u_o^2} S(u, |u_o|, z)$$

in order to better obtain a symmetric form yields

$$\frac{dS}{dz}(u, |u_o|, z) + \frac{S}{|u_o|}(u, |u_o|, z) + \frac{S}{u}(u, |u_o|, z)$$

$$= 1 + \frac{1}{\pi}\int_o^\infty \frac{e^{-u'^2}}{u'} S(u', |u_o|, z)\, du'$$

$$+ \frac{1}{\pi}\int_o^\infty \frac{e^{-u_o'^2}}{|u_o'|} S(u, |u_o'|, z)\, du_o'$$

$$+ \frac{1}{\pi}\int_o^\infty\int_o^\infty \frac{e^{-u_o'^2}\, e^{-u'^2}}{|u_o'| \cdot u'} S(u, |u_o'|, z) S(u', |u_o|, z)\, du'\, du_o' \qquad (13)$$

This equation was derived in Ref. 1 by particle-counting techniques.

METHOD II

A direct and powerful approach to the development of the invariant-imbedding equations from the Boltzmann formulation uses simple perturbation techniques as the principal tool. We apply it here to a discrete model of the Boltzmann equation and then indicate the results for continuous models.

Discrete Model

We introduce a discrete model of the Boltzmann equation:

$$\frac{df_i^+}{dx} = K_i^+(f_1^-, f_2^- \ldots f_N^-, f_1^+, f_2^+ \ldots f_N^+)$$

$$\frac{df_i^-}{dx} = -K_i^-(f_1^+, f_2^+, \ldots f_N^+, f_1^-, f_2^- \ldots f_N^-) \qquad (14)$$

The $K_i^{\pm}$'s are related to discrete versions of the collision operator and are not restricted solely to linear operators on the $f_i^{\pm}$'s. Discrete models of transport equations have been considered for many years in neutron and radiative transfer, but it is only in recent years that attempts have been made to utilize them in kinetic theory. We note that the subscript i refers to a particular value of the velocity vector, $\vec{C}_i$, defined by $\vec{C}_i = (u_{i_u}, v_{i_v}, w_{i_w})$, the f_i^+ refers to the distribution function of molecules having positive u components, and the f_i^-'s refer to molecules having negative u components where u is positive in the direction of increasing x.

The discrete model arises from the consideration of a discrete set of molecular velocities, $\vec{C}_i$, rather than a continuous set. Finite volumes in velocity space $\Delta\vec{C}_i = \left(\Delta u_{i_u}\right)\left(\Delta v_{i_v}\right)\left(\Delta w_{i_w}\right)$ are utilized, rather than infinitesmal elements $d\vec{c} = du\, dv\, dw$; integrals are replaced by sums, and a distribution function corresponding to each of the states $\vec{C}_i$ is sought.

We consider boundary conditions resulting from an arbitrary source of molecules at x = z and a sink for molecules at x = 0. A somewhat improbable physical interpretation of these boundary conditions is that the wall at x = z evaporates molecules into the flow and possesses a unit sticking coefficient, while the wall at x = 0 condenses all molecules which impinge on it. For linear problems, as discussed earlier, this standard set of boundary conditions is sufficient to account for other more interesting surface-particle interactions.

These boundary conditions are

$$\begin{aligned} x &= 0 \qquad f_i^+ = 0 \\ x &= z \qquad f_i^- = \gamma_i^- \end{aligned} \tag{15}$$

Let the solution of Eq. (14) on the interval 0, z, subject to the source γ_i^- at x = z, be indicated by f_i^+, f_i^-.

Similarly, let the solution of Eq. (15) on the interval 0, z + Δ, subject to the source γ_i^- at x = z + Δ, be indicated by F_i^+, F_i^-.

Thus we may write

$$\frac{df_i^+}{dx} = k_i^+(f_1^+, f_2^+ \ldots f_N^+, f_1^-, f_2^- \ldots f_N^-)$$

$$\frac{df_i^-}{dx} = -k_i^-(f_1^+, f_2^+ \ldots f_N^+, f_1^-, f_2^- \ldots f_N^-) \tag{16}$$

where i runs from 1 to N with boundary conditions

$$f_i^+(0) = 0$$

$$f_i^-(z) = \gamma_i^- \tag{17}$$

and

$$\frac{dF_i^+}{dx} = K_i^+(F_1, F_2 \ldots F_N, F_1^-, F_2^- \ldots F_N^-)$$

$$\frac{dF_i^-}{dx} = -K_i^-(F_1^+, F_2^+ \ldots F_N^+, F_1^-, F_2^- \ldots F_N^-) \tag{18}$$

where the $K_i^{\pm}$ is the same collision operator as the $k_i^{\pm}$, where

$$F_i^+(0) = 0$$

$$F_i^-(z + \Delta) = \gamma_i^- \tag{19}$$

We seek perturbation solutions to Eq. (18) in the form

$$F_i^+ = f_i + \Delta \cdot r_i^+ + o(\Delta)$$

$$F_i^- = f_i^- + \Delta \cdot w_i^- + o(\Delta) \tag{20}$$

The boundary conditions on r_i^+, w_i^- can be obtained as follows:

$$r_i^+(0) = 0 \qquad \text{at } x = 0 \tag{21a}$$

At x = z, we expand in a Taylor series to obtain

$$F_i^-(z) = F_i^-(z + \Delta) - \Delta \cdot \frac{\partial F_i^-}{\partial x}(z + \Delta) + o(\Delta) \tag{21b}$$

or

$$f_i^-(z) + \Delta \cdot w_i^-(z) = \gamma_i^- - \Delta \cdot \frac{\partial f_i^-}{\partial x}(z) + o(\Delta)$$

or

$$w_i^-(z) = k_i^-(f_1^+(z),\ f_2^+(z)\ \ldots\ f_N^+(z),\ f_1^-(z),\ f_2^-(z)\ \ldots\ f_N^-(z)) \tag{21c}$$

The linear differential equations for the perturbations r_i^+, w_i^- are obtained by substitution of Eq. (20) into Eq. (18).

$$\begin{aligned} \frac{\partial r_i^+}{dx} &= \sum_{j=1}^{N} \frac{\partial k_i^+}{\partial f_j^+} r_j^+ + \sum_{j=1}^{N} \frac{\partial k_i^+}{\partial f_j^-} w_j^- \\ \frac{\partial w_i^-}{dx} &= -\sum_{j=1}^{N} \frac{\partial k_i^-}{\partial f_j} r_j^+ + \sum_{j=1}^{N} \frac{\partial k_i^-}{\partial f_j^-} w_j^- \end{aligned} \tag{22}$$

The solution to this set of equations, subject to boundary conditions in Eqs. (21), is

$$r_i^+(x) = \sum_{j=1}^{N} \frac{\partial f_i}{\partial \gamma_j^-} \cdot k_j^-(f_1^+(z),\ f_2^+(z)\ \ldots\ f_N^+(z),\ f_1^-(z),\ f_2^-(z)\ \ldots\ f_N^-(z))$$

$$w_i^-(x) = \sum_{j=1}^{N} \frac{\partial f_i^-}{\partial \gamma_j^-} k_j^-(f_1^+(z),\ f_2^+(z)\ \ldots\ f_N^+(z),\ f_1^-(z),\ t_2^-(z)\ \ldots\ F_N^-(z)) \tag{23}$$

which can be verified by substitution.

At $x = z + \Delta$, a Taylor series expansion results in

$$F_i^+(z + \Delta) = F_i^+(z) + \frac{\Delta \partial F_i^+(z)}{\partial x} + o(\Delta)$$

$$= f_i^+(z) + \Delta r_i^+(z)$$

$$+ \Delta k_i^+(f_1^+(z),\ f_2^+(z)\ \ldots\ f_N^+(z),\ f_1^-(z),\ f_2^-(z)\ \ldots\ f_N^-(z)) + o(\Delta) \tag{24}$$

At this point in the analysis it is appropriate to introduce a reflection function that must differ essentially from the reflection function of the linear problem considered earlier. In the linear problem, it was assumed that the distribution function for particles impinging on the boundary at $x = z$ was a linear functional of the distribution function for particles emitted from that boundary. The definition of the reflection function in Eqs. (3) and all the succeeding analysis was engendered by the linearity of the collision operator. Thus it was possible to consider a delta-function incident flux of molecules and construct from this the solution for any incident flux. In this nonlinear version of the same problem it is no longer possible to uncouple the e[illegible] of a particular source distribution at the boundary at $x = z$ from the resulting flux of particles incident upon that boundary. We introduce the function $R_i^*(z, \gamma_1^-, \gamma_2^- \cdots \gamma_N^-)$ which is the distribution function of particles incident upon the boundary

at $x = z$ due to the particular distribution of incident particles at $x = z$ emitted by that boundary, $\gamma_1^-, \gamma_2^- \cdots \gamma_N^-$.

Thus

$$R_i^*(z, \gamma_1^-, \gamma_2^- \cdots \gamma_N^-) \equiv f_i^+(z,)$$

$$R_i^*(z + \Delta, \gamma_1^-, \gamma_2^- \cdots \gamma_N^-) \equiv F_i^+(z + \Delta)$$

and

$$\frac{\partial R_i^*}{\partial z} = \lim_{\Delta \to 0} \frac{F_i(z + \Delta) - f_i(z)}{\Delta}$$

The resulting equation for $R_i^*(z)$ is then

$$\frac{\partial R_i^*}{\partial z}(z, \gamma_1^-, \gamma_2^- \cdots \gamma_N^-) = \sum_{j=1}^{N} \frac{\partial R_i(x, \gamma_1^-, \gamma_2^- \cdots \gamma_N^-)}{\partial \gamma_j^-}$$

$$\times\, k_j^-(R_1^*(z), R_2^*(z) \ldots R_N^*(z), \gamma_1^-, \gamma_2^- \cdots \gamma_N^-)$$

$$+\, k_i^+(R_1(z), R_2(z) \ldots R_N(z), \gamma_1^-, \gamma_2^- \cdots \gamma_N^-)$$

(25)

Continuous Model

If a continuous version of Eqs. (16) and (17) is considered,

$$\frac{df^+}{dx}(\vec{C}^+, x) = K^+(\vec{C}^+)$$

$$\frac{df^-}{dx}(\vec{C}^-, x) = -K^-(\vec{C}^+) \tag{26}$$

$$f^+(\vec{C}^+, 0) = 0$$

$$f^-(\vec{C}^-, z) = \gamma(\vec{C}^-)$$

where the $K(\vec{C}^\pm)$ are functionals of the distribution functions $f^\pm$, and $\vec{C}^-$ takes on all values of the velocity vector possessing a component in the direction of decreasing x, while $\vec{C}^+$ takes on all values possessing a component in the direction of increasing x. The results of the previous section may be extended formally to give

$$\frac{\partial R^*}{\partial z}(\gamma(\vec{C}), \vec{C}, z) = \int_{\vec{C}^-} \frac{\delta R^*}{\delta \gamma}(\gamma(\vec{\xi}), \vec{C}, z) K^-(\vec{\xi}^-, z)\, d\vec{\xi}^- + K^+(\vec{C}^+, z) \tag{27}$$

where $\frac{\partial R^*}{\delta \gamma}$ is the functional partial derivative, defined by

$$\lim_{\Delta \to 0} \frac{R^*(\gamma(\vec{C}) + \Delta K^-(\vec{C}), \vec{C}, z) - R^*(\gamma(\vec{C}), \vec{C}, z)}{\Delta}$$

$$= \int_{-\infty}^{0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R^*(\gamma(\vec{C}_o), \vec{C}, z) \cdot K^-(\vec{C}_o, z)\, d\vec{C}_o \tag{28}$$

If K is a linearized collision operator, we introduce a reflection function $R^*(u, v, w, u_o, v_o\, w_o, z)$ such that

$$R^*(\gamma(\vec{C}_o), \vec{C}, z) = \int_{-\infty}^{0} \int_{+\infty}^{-\infty} \int_{-\infty}^{+\infty} R^*(u, v, w, u_o, v_o, w_o, z)$$

$$\times\ \gamma(u_o, v_o, w_o)\ du_o\ dv_o\ dw_o \qquad (29)$$

and note that

$$\frac{\delta R^*}{\delta \gamma} = R^*(u, v, w, u_o, v_o, w_o, z) \qquad (30)$$

Substituting Eqs. (29) and (30) in Eq. (27), we recover the equivalent of Eq. (9):

$$\frac{\partial}{\partial z} = \int_{-\infty}^{0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R^*(u, v, w, u_o, v_o, w_o, z)\gamma(u_o, v_o, w_o)\ du_o\ dv_o\ dw_o$$

$$= - \int_{-\infty}^{0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R^*(u, v, w, u_o', v_o', w_o', z)\ \frac{L^-}{u_o'}$$

$$\times\ (u_o', v_o', w_o', z)\ du_o'\ dv_o'\ dw_o' + \frac{1}{u} L^+(u, v, w, z) \qquad (31)$$

The initial condition is always $R'(\vec{C}, \vec{C}_o, 0) \equiv 0$, stemming from the observation that there are no collisions to reflect particles from $\vec{C}_o$ to $\vec{C}$ if $\tau = 0$.

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10. ABSTRACT

A re-examination of the application of invariance techniques to plane Couette-flow problems and a presentation of several different approaches to the derivation of the invariant-imbedding equations. It is shown how both linear and nonlinear collision operators may be considered, for both discrete and continuous versions of the original Boltzmann formulation.

11. KEY WORDS

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